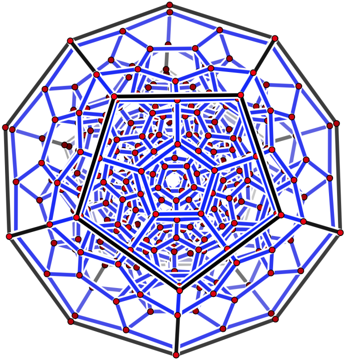
Polytopes:
Extremal Examples and Combinatorial Parameters

Günter M. Ziegler



Outline

Before I start

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- EXAMPLES: rich theory! (write a "Book of Examples"!?)
- PROBLEMS: wonderful conjectures, challenges, things to do!

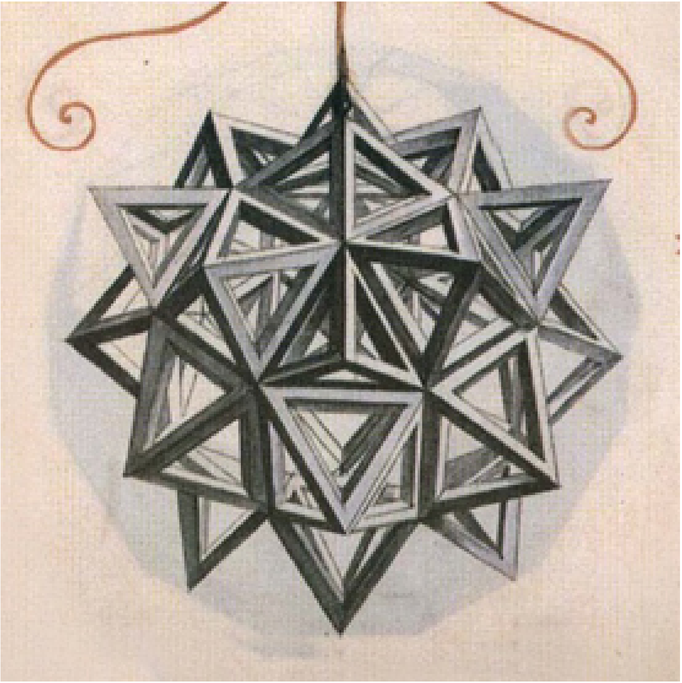


Image: Leonardo da Vinci drawing for Pacioli's book "De Divina Proportione"

Before I start

"It is not unusual that a single example or a very few shape an entire mathematical discipline. Examples are the Petersen graph, cyclic polytopes, the Fano plane, the prisoner dilemma, the real n-dimensional projective space and the group of two by two nonsingular matrices. And it seems that overall, we are short of examples."

— Gil Kalai 2000: "Combinatorics with a Geometric Flavor"

Lecture 1:
3-Dimensional Polytopes

Definition

Definition (3-Dimensional Polytope)

A *3-dimensional polytope* is the convex hull of a finite set of points, which do not all lie on a plane:

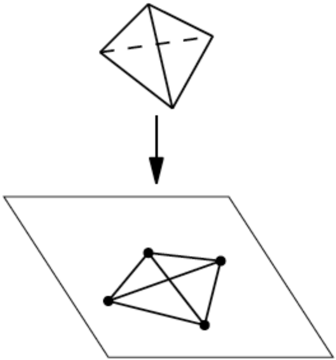
For $v_1, \ldots, v_n \in \mathbb{R}^3$:

$$\mathrm{conv}\{v_1, \ldots, v_n\} \;:=\; \{x_1 v_1 + \cdots + x_n v_n \in \mathbb{R}^3 : \; x_1 + \cdots + x_n = 1, \\ x_0, \ldots, x_n \geq 0 \;\}$$

Definition

Equivalently, any 3-polytope with n vertices is "by definition" a linear image of the $(n-1)$-dimensional simplex

$$\Delta_{n-1} \quad := \quad \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1, \\ x_0, \ldots, x_n \geq 0 \}.$$



Faces

Definition (Faces: vertices, edges, facets)

A *face* of a polytope consists of all points that maximize a linear function.

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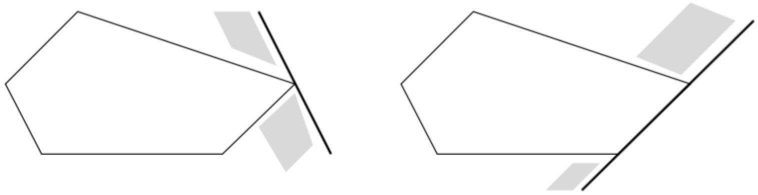
Each face is itself a polytope (of smaller dimension).

0-dimensional faces are called *vertices*,
1-dimensional faces are called *edges*,
$(d-1)$-dimensional faces are called *facets*.



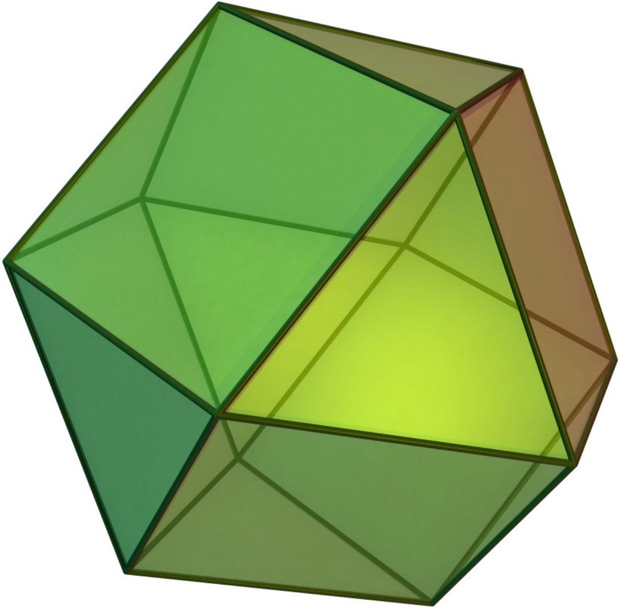


Image: Wikipedia

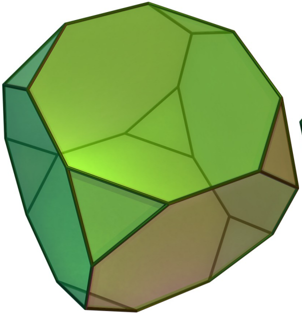
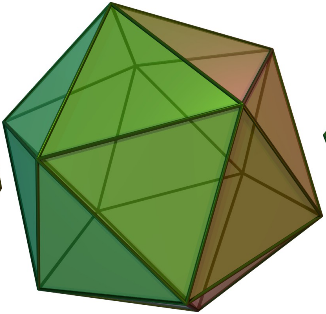
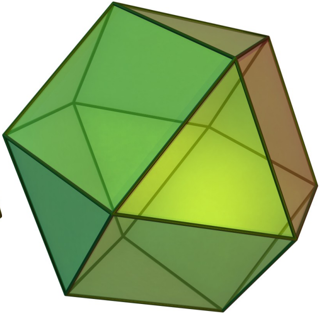
Simple/simplicial polytopes

Definition

A 3-polytope is *simplicial* if all its 2-faces are triangles.
A 3-polytope is *simple* if all its vertices have degree 3.

(We do not talk much here about *duality*, but this exists, and is important, and the dual of any simple polytope is simplicial, and vice versa.)

Examples:
Truncated Hexahedron (cube), Icosahedron, and Cuboctahedron

Images: Wikipedia

The f-vector

Definition

For a 3-polytope P the *f-vector* is $f(P) = (f_0, f_1, f_2)$ with

$$f_i := \#i\text{-dimensional faces of } P.$$

The f-vector

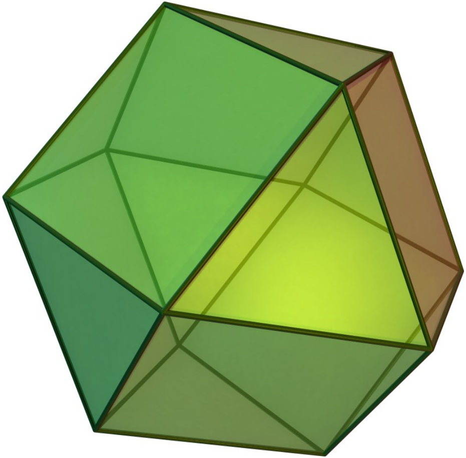


Image: Wikipedia

$f_0 =$

The f-vector

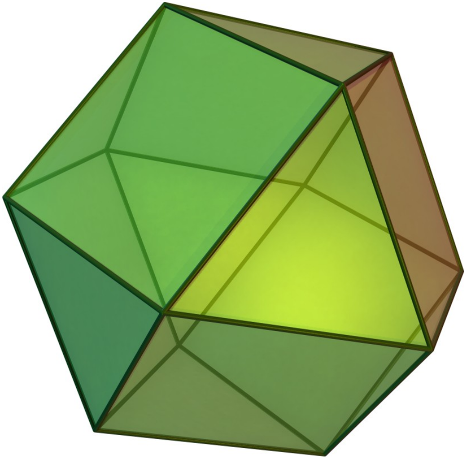


Image: Wikipedia

$f_0 = 12$

$f_1 =$

The f-vector

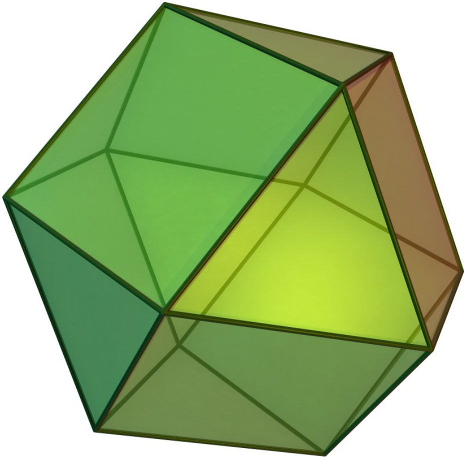


Image: Wikipedia

$f_0 = 12$
$f_1 = 24$
$f_2 =$

The f-vector

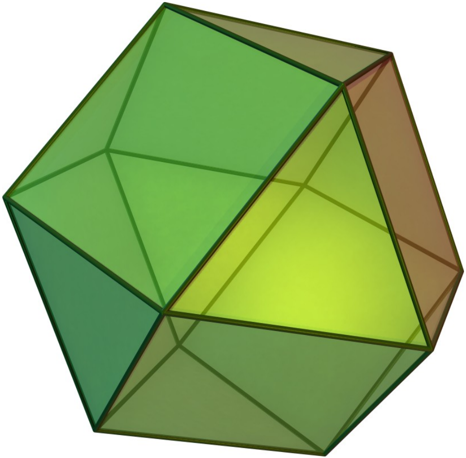


Image: Wikipedia

$f_0 = 12$
$f_1 = 24$
$f_2 = 14$

f-vector:

The f-vector

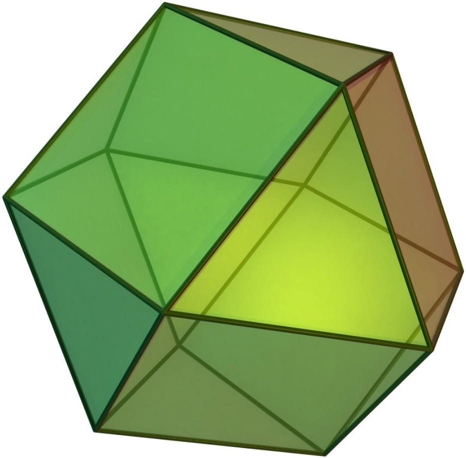


Image: Wikipedia

$f_0 = 12$
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$f_2 = 14$

f-vector: $(12, 24, 14)$

Euler's Equation

Proposition (Euler's Equation)

The face numbers of any 3-polytope satisfy

$$f_0 - f_1 + f_2 = 2.$$

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Proof.

There are 20 of them! Do it yourself! □

The Geometry Junkyard

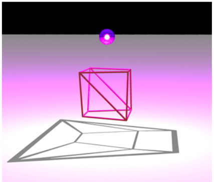
Twenty Proofs of Euler's Formula: V-E+F=2

Many theorems in mathematics are important enough that they have been proved repeatedly in surprisingly many different ways. Examples of this include the existence of infinitely many prime numbers, the evaluation of zeta(2), the fundamental theorem of algebra (polynomials have roots), quadratic reciprocity (a formula for testing whether an arithmetic progression contains a square) and the Pythagorean theorem (which according to Wells has at least 367 proofs). This also sometimes happens for unimportant theorems, such as the fact that in any rectangle dissected into smaller rectangles, if each smaller rectangle has integer width or height, so does the large one.

This page lists proofs of the Euler formula: for any convex polyhedron, the number of vertices and faces together is exactly two more than the number of edges. Symbolically V−E+F=2. For instance, a tetrahedron has four vertices, four faces, and six edges; 4-6+4=2.

A version of the formula dates over 100 years earlier than Euler, to Descartes in 1630. Descartes gives a discrete form of the Gauss-Bonnet theorem, stating that the sum of the face angles of a polyhedron is $2\pi(V-2)$, from which he infers that the number of plane angles is 2F+2V-4. The number of plane angles is always twice the number of edges, so this is equivalent to Euler's formula, but later authors such as Lakatos, Malkevitch, and Polya disagree, feeling that the distinction between face angles and edges is too large for this to be viewed as the same formula. The formula V−E+F=2 was (re)discovered by Euler; he wrote about it twice in 1750, and in 1752 published the result, with a faulty proof by induction for triangulated polyhedra based on removing a vertex and retriangulating the hole formed by its removal. The retriangulation step does not necessarily preserve the convexity or planarity of the resulting shape, so the induction does not go through. Another early attempt at a proof, by Meister in 1784, is essentially the triangle removal proof given here, but without justifying the existence of a triangle to remove. In 1794, Legendre provided a complete proof, using spherical angles. Cauchy got into the act in 1811, citing Legendre and adding incomplete proofs based on triangle removal, ear decomposition, and tetrahedron removal from a tetrahedralization of a partition of the polyhedron into smaller polyhedra. Hilton and Pederson provide more references as well as entertaining speculation on Euler's discovery of the formula. Confusingly, other equations such as $e^{i\ \mathrm{pi}} = -1$ and $a^{\mathrm{phi}(n)} = 1 \pmod{n}$ also go by the name of "Euler's formula"; Euler was a busy man.

The polyhedron formula, of course, can be generalized in many important ways, some using methods described below. One important generalization is to planar graphs. To form a planar graph from a polyhedron, place a light source near one face of the polyhedron, and a plane on the other side.



The Upper Bound Theorem

Corollary (Upper Bound Theorem, 3D version)

The face numbers of any 3-polytope satisfy

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with equality if P is simplicial.
Combine this with Euler's equation:
$3f_2 \leq 2f_1 = 2(f_0 + f_2 - 2) = 2f_0 + 2f_2 - 4.$ □

The f-vectors of 3-polytopes (Steinitz 1906)

Theorem

The set of f-vectors of 3-dimensional polytopes is the set of all integer points in a 2-dimensional cone:

$$
\begin{aligned}
f(\mathcal{P}^3) \quad = \quad \{(f_0, f_1, f_2) \in \mathbb{Z}^3 : \quad & f_0 - f_1 + f_2 = 2, \\
& f_2 \leq 2f_0 - 4, \\
& f_0 \leq 2f_2 - 4 \ \}.
\end{aligned}
$$

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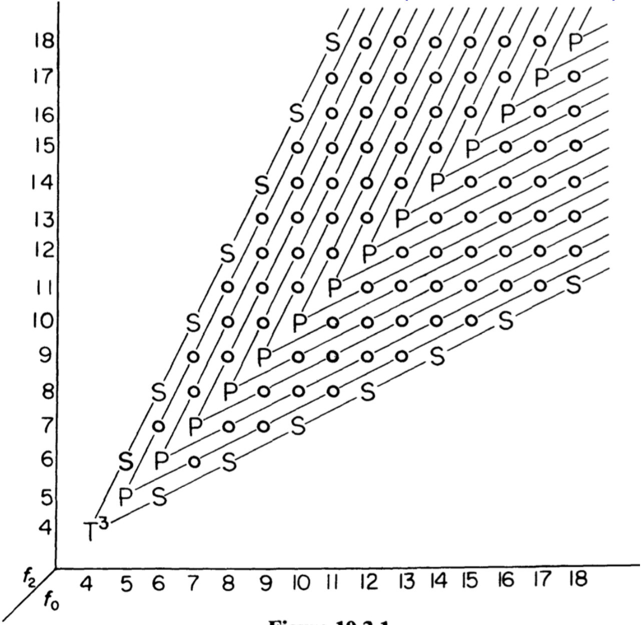

Figure 10.3.1

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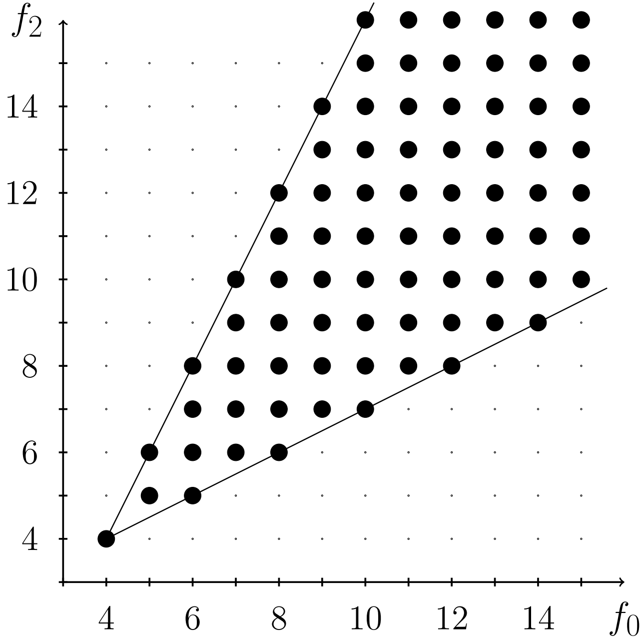

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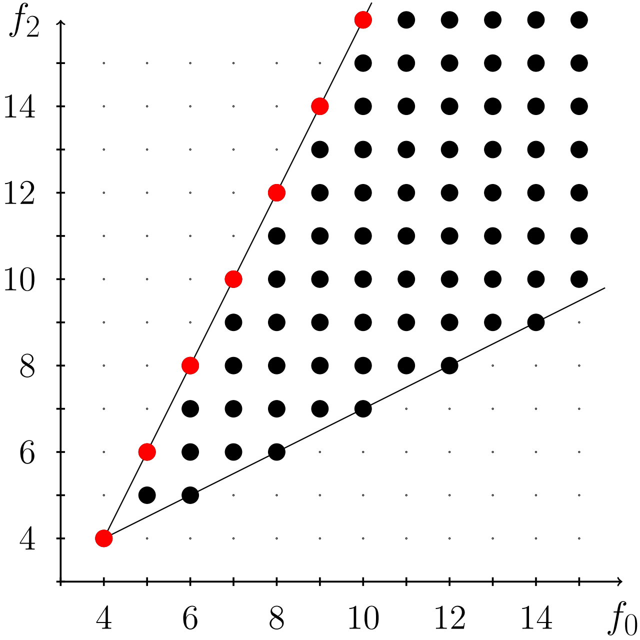

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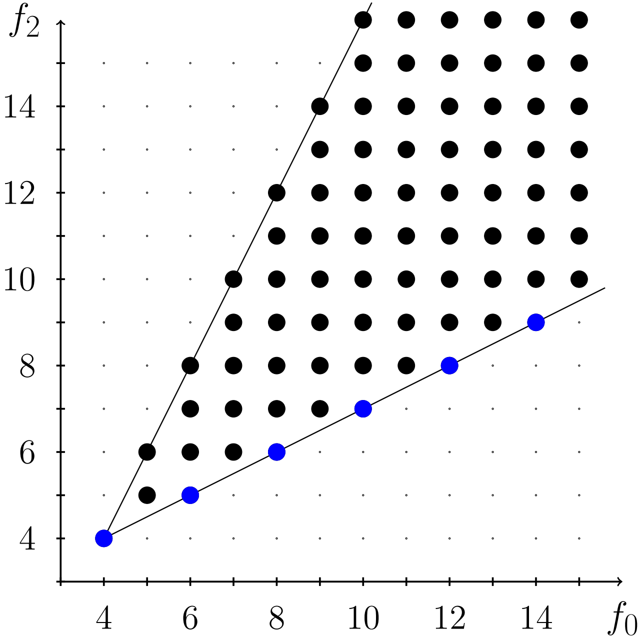

The Face Lattice — The Combinatorial Type

Definition (The Face Lattice)

The set of all (!) faces of a polytope
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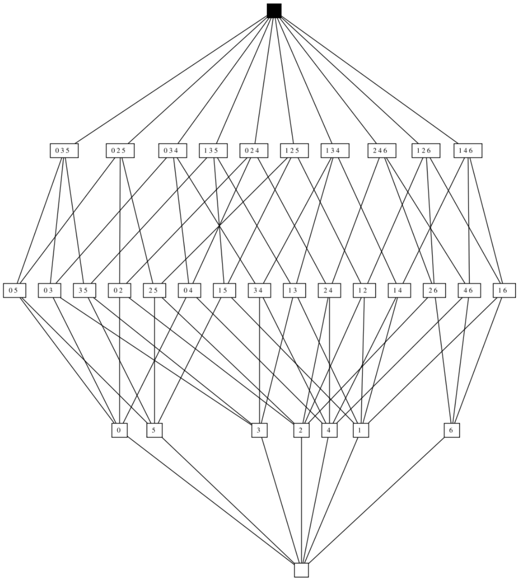
The face lattice (as an abstract partially ordered set)
collects all the combinatorial information:

Definition (Combinatorially Equivalent)

Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic.

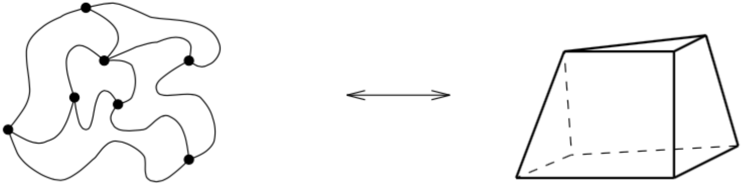
The Face Lattice — The Combinatorial Type



Steinitz's Theorem [Ernst Steinitz 1922]

Theorem (Steinitz's Theorem)

There is a bijection between 3-connected planar graphs and combinatorial types of 3-dimensional polytopes.



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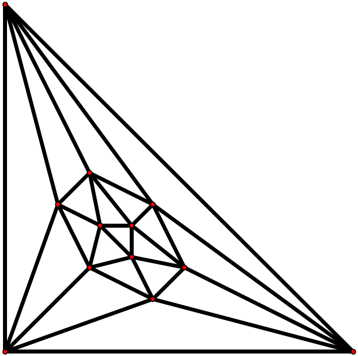

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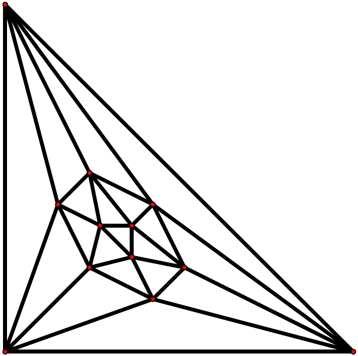
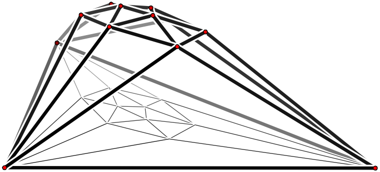
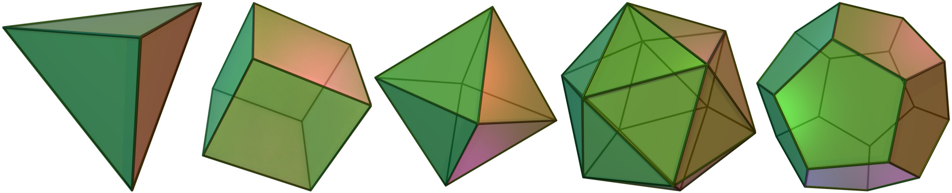




The Platonic Solids

Theorem (Euclid?: Classification of the Platonic Solids)

There are exactly five (similarity types of) 3D regular polytopes:



Images: Wikipedia

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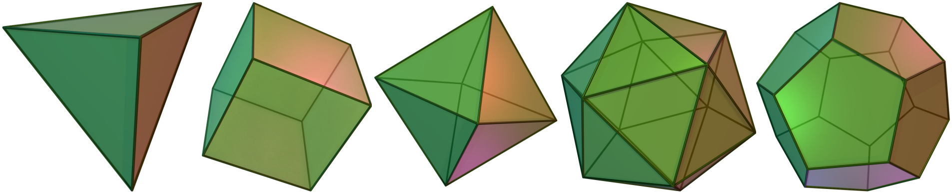

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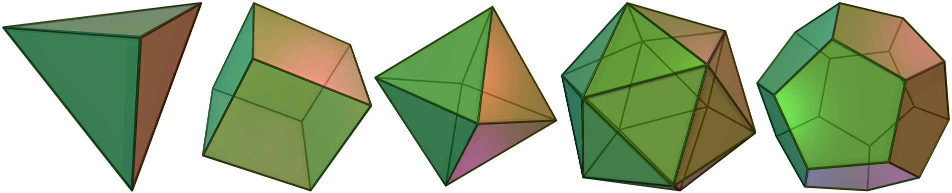


Images: Wikipedia

Proof.

Necessity part (this is the only 5 possibly types): do it!
Sufficiency part (they exist): see the exercise sheet! □

The Archimedian Solids

Theorem (Kepler?: Classification of the Archimedian Solids)

There are exactly 13 (similarity types of) 3D Archimedian polytopes:

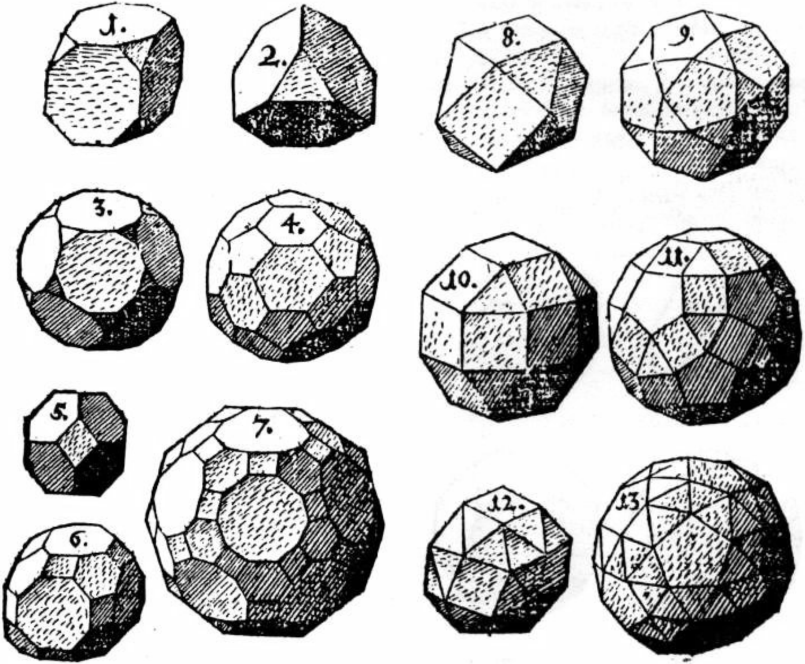


Image: Kepler 1609

EXAMPLE: The pseudo-rhombicuboctahedron

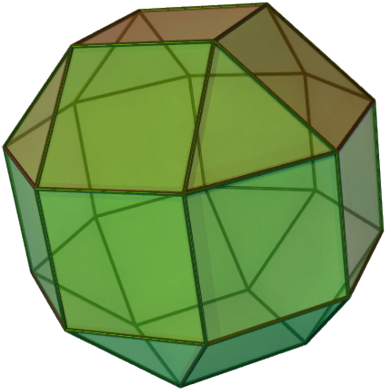


Image: Wikipedia

"Miller solid" or "Johnson body J_{37}"

EXAMPLE: The pseudo-rhombicuboctahedron

Sommerville (1906)

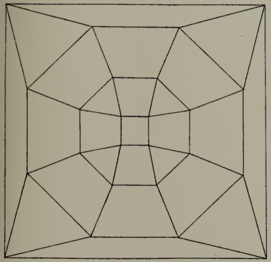


Fig. 26.

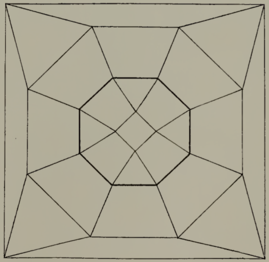


Fig. 27.

EXAMPLE: The pseudo-rhombicuboctahedron

OPEN PROBLEM:

Complete and correct write-up of the classification of the Archimedian solids, including

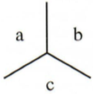
— combinatorial classification

— existence

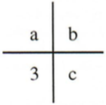
— uniqueness

this result which applies to vertices surrounded by four polygons can be used to exclude the case (3,3,4,4) above. These two results are collected together in the following lemma.

Lemma. A polyhedron in which all the solid angles are surrounded in the same way cannot have solid angles of the following types:

(*i*) 

where a is odd and $b \neq c$.

(*ii*) 

where $a \neq c$.

PROOF: In the first case, the fact that all the solid angles have the same type implies that the b-gon faces must alternate with the c-gon faces round the boundary of an a-gon face. But, since a is assumed to be odd, this leads to a contradiction. This is clearly seen in the example shown in Figure 4.14(a) which illustrates the case when $a = 7$.

In case (*ii*), we consider the way that the faces must be arranged around the 3-gon. At each angle, the face opposite the 3-gon is always a b-gon. Since all the vertices have the same type, the sides of the 3-gon must be attached to a-gons and c-gons, and these must alternate around the 3-gon. This again leads to an inconsistency (see Figure 4.14(b)). ■

[Cromwell 1997]

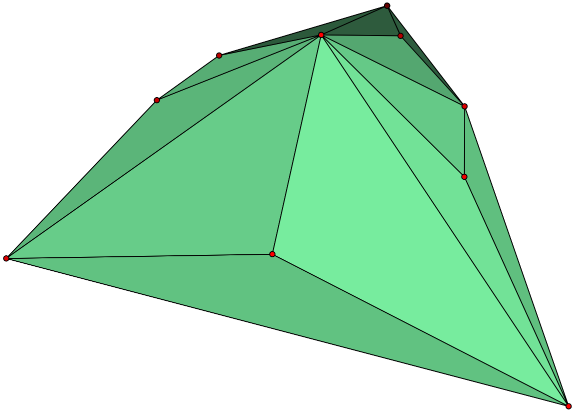
OPEN: Small coordinates?

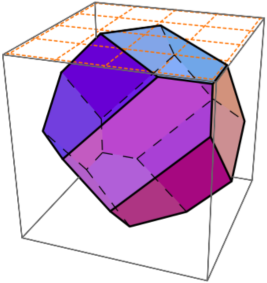
Problem

Can one realize all 3-polytopes with polynomial size integer vertex coordinates?
That is, can all n-vertex polytopes be realized with their vertices in $\{0, 1, \ldots, n^K\}^3$*, for some* K*?*

OPEN: Small coordinates?

Even for stacked polytopes, this is hard to prove:
see [Demaine & Schulz, DCG 2017]





The coordinates of the vertices are:

$$\pm\{(2,2,2),(2,2,1),(2,1,2),(1,2,2),$$
$$(2,-1,0),(2,0,-1),(-1,2,0),$$
$$(0,2,-1),(0,-1,2),(-1,0,2)\}$$

Fig. 3. The smallest embedding of the dodecahedral graph as a convex polyhedron

[Igamberdiev, Nielsen & Schulz 2013]

Lecture 2:

The d-Cubes and the Hypersimplices

What is a polytope?

Definition (V-Polytope, H-Polytope)

A *V-Polytope* is the convex hull $P = \mathrm{conv}(V)$ of a finite set of points $V \subset \mathbb{R}^d$.

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An *H-Polytope* is the solution set $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ of a finite set of linear inequalities
— provided that this solution set is bounded.

V=H

Theorem (Weyl, Minkowski: V=H)

Every V-polytope is an H-polytope, and conversely.

V=H

Theorem (Weyl, Minkowski: V=H)

Every V-polytope is an H-polytope, and conversely.

This can be proved by

1. writing $\mathrm{conv}(V)$ as linear image of Δ_{n-1},
2. describing Δ_{n-1} by linear inequalities,
3. showing that "can be described by linear inequalities" is preserved by "project down by one dimension."

V=H

Theorem (Weyl, Minkowski: V=H)

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The converse direction is proved similarly – or by using duality.

Example: The d-cube

Definition (The d-cube)

The d-cube can be defined as

$$\begin{aligned} C_d &:= \operatorname{conv}\{-1,+1\}^d \\ &= \{x \in \mathbb{R}^d : -1 \le x_i \le +1 \text{ for all } i\}. \end{aligned}$$

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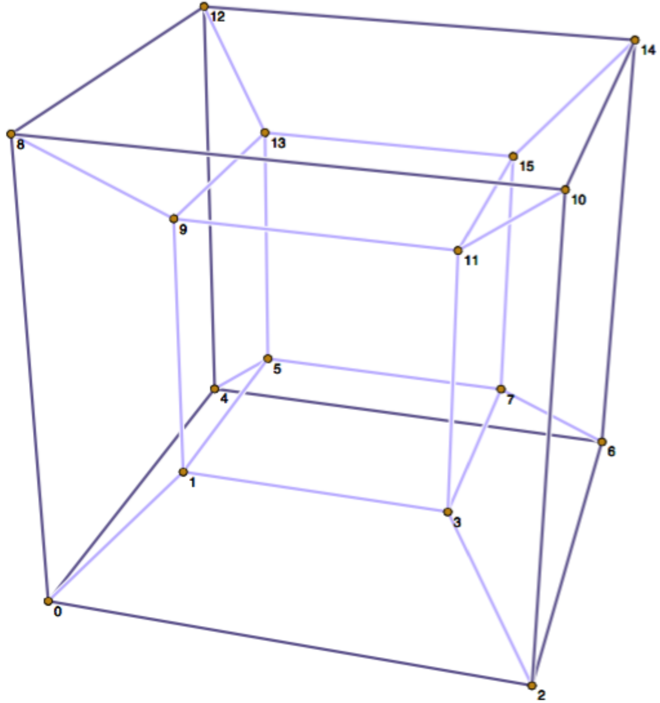
Similarly the d-dimensional octahedron ("cross polytope")
has few ($2d$) vertices and many (2^d) facets!



12
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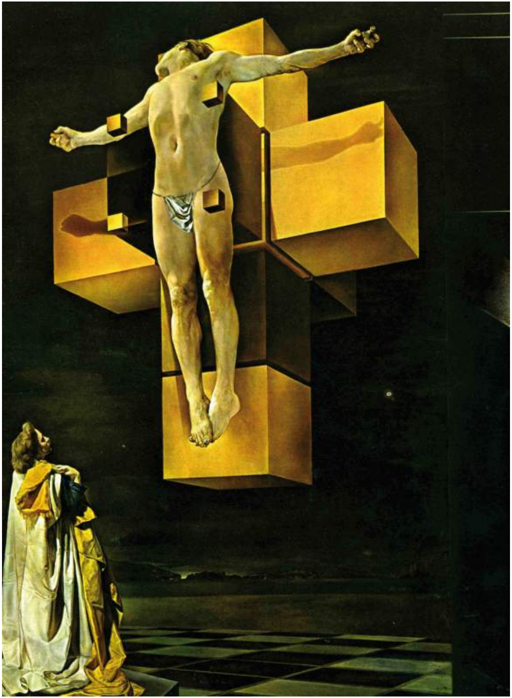

Image: Crucifixion (Corpus Hypercubus), by S. Dali 1954

Image: "Robocopus Hypercubus" by Tim Doyle

The convex hull problem

OPEN PROBLEM:
Given the vertices of a polytope,
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The convex hull problem

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OPEN PROBLEM:
Given a set of points and a set of inequalities,
can you tell in polynomial time whether they describe the same polytope?

Faces and the face lattice

Definition: faces, f-vector, face lattice

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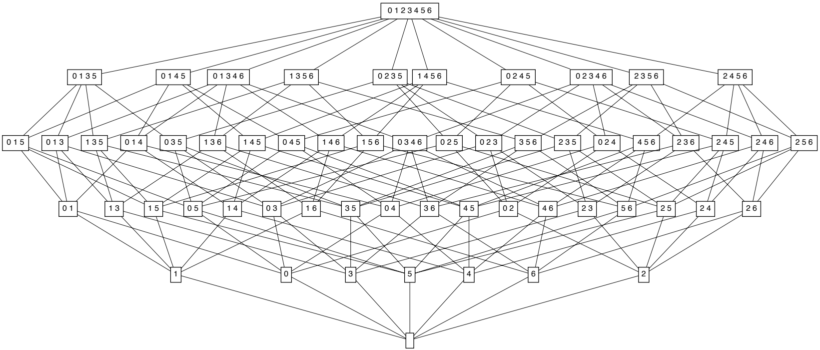
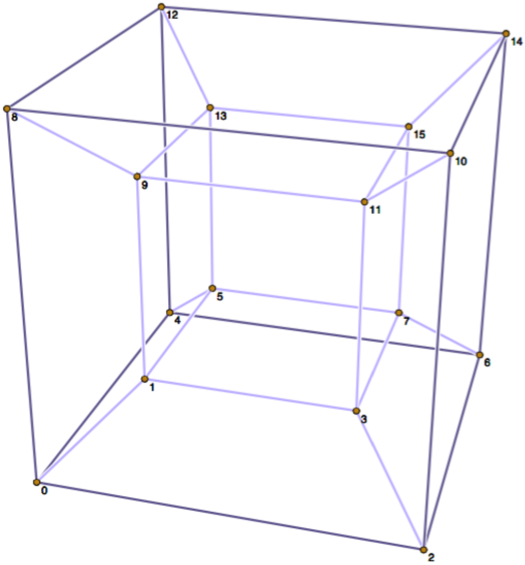
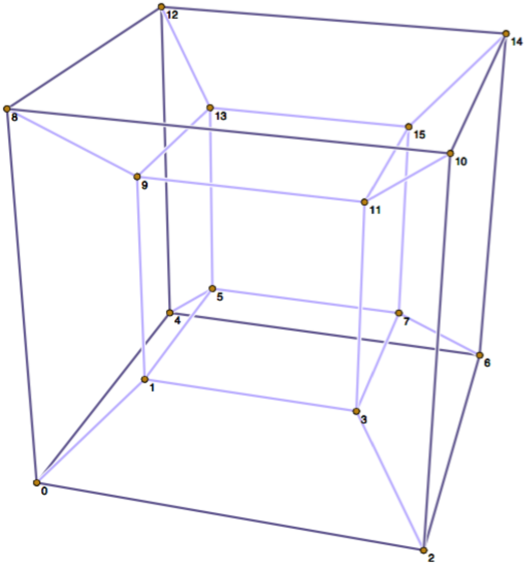




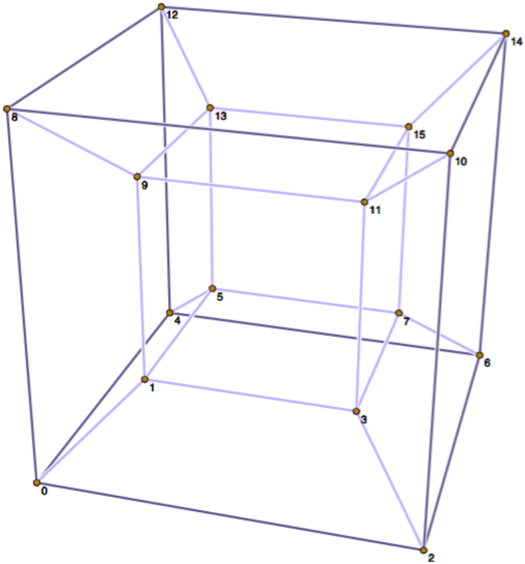
The f-vector of the 4-cube is

$$f(C_4) = (\quad , \quad , \quad , \quad).$$



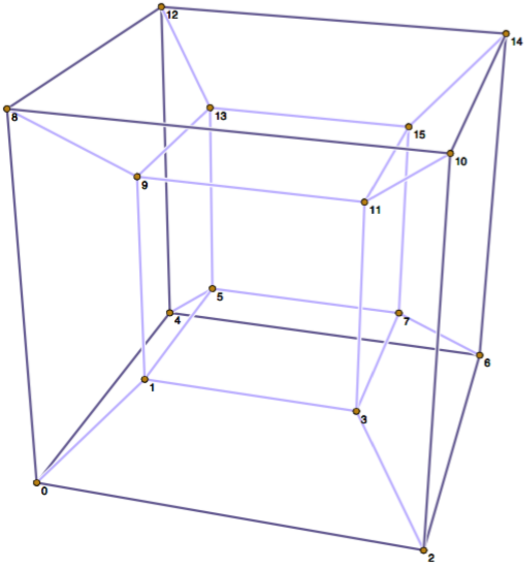
The f-vector of the 4-cube is

$$f(C_4) = (16, \quad , \quad , \quad).$$



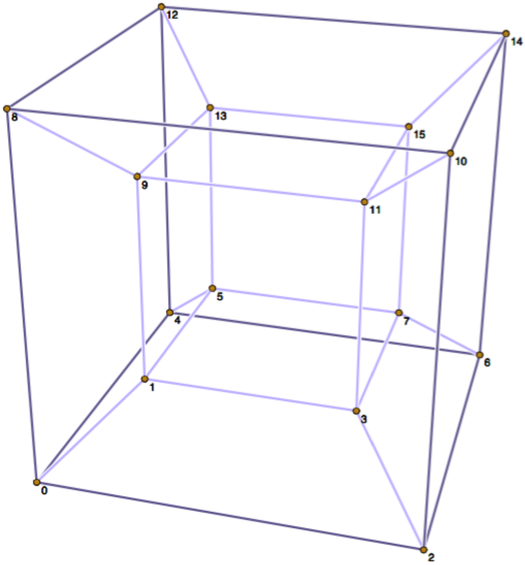
The f-vector of the 4-cube is

$$f(C_4) = (16, 32, \quad , \quad).$$



The f-vector of the 4-cube is

$$f(C_4) = (16,\ 32,\ 24,\ \quad).$$



The f-vector of the 4-cube is

$$f(C_4) = (16,\ 32,\ 24,\ 8).$$

Example: The d-cube

Theorem (The f-vector of the d-cube)

$f_0 = 2^d$,

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Note that the d-cube is simple — this can be seen from $2f_1 = df_0$.

The Euler-Poincaré equation

Theorem (Euler-Poincaré equation)

For every d-dimensional polytope,

$$f_0 - f_1 + f_2 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d.$$

Proof.

Shelling! [Schläfli ca. 1850] [Brugesser-Mani 1970]
Homology [Poincaré ca. 1905] □

The Hypersimplices

Definition (The Hypersimplices – two versions)

For $1 \leq k \leq d$, the d-dimensional *hypersimplices* $\Delta_d(k)$ and $\Delta'_d(k)$ are given by

$$\Delta_d(k) \quad := \quad \{x \in [0,1]^{d+1} : \qquad \sum_{i=1}^{d+1} x_i = k\}$$

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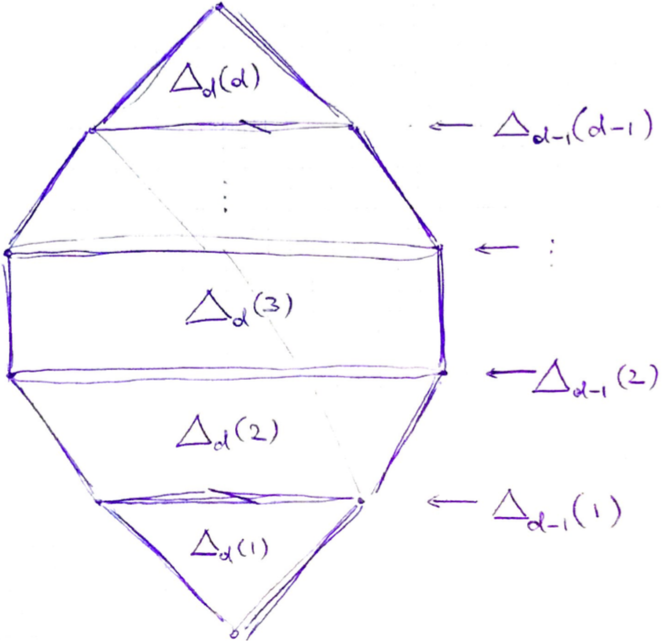
The Hypersimplices



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4. *All vertices of* $\Delta_d(k)$ *are equivalent under symmetry.*
5. $\Delta_d(k)$ *has* $2(d+1)$ *facets, of two types:*
   $d+1$ *of type* $\Delta_{d-1}(k)$
   $d+1$ *of type* $\Delta_{d-1}(k-1)$

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6. $\Delta_d(\frac{d+1}{2})$ *is centrally symmetric (for odd* d*)*

EXAMPLE: “The hypersimplex” $\Delta_4(2)$

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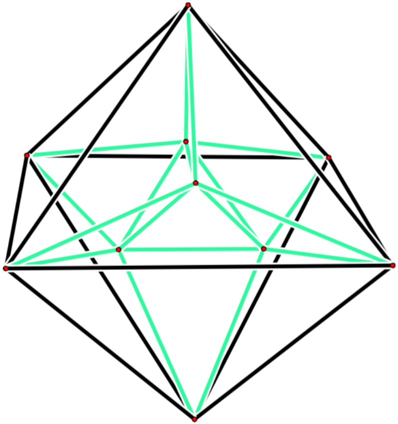

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Definition/Lemma: $\Delta_4(2)$ is 2-simple 2-simplicial

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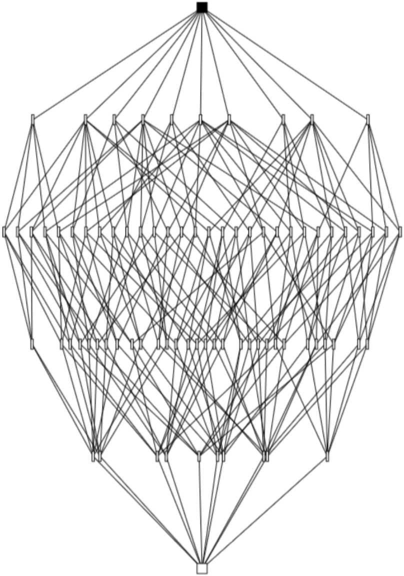

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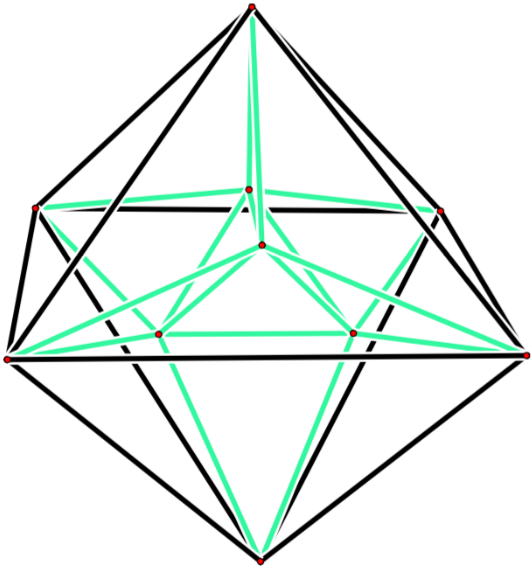

EXAMPLE: "The hypersimplex" $\Delta_4(2)$

OPEN PROBLEM (The "fatness problem"):
For a 4-polytope with $f_0 = f_3 = n$, how large can $f_1 = f_2$ be?

Lecture 3:

Extremal Polytopes,

Extremal f-Vectors

Simplicial polytopes

Definition

A d-dimensional polytope is *simplicial* if all its facets are simplices, that is, they have exactly d vertices.

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For any fixed $n > d$, some simplicial polytope maximizes all face numbers among all d-polytopes with n vertices.

Proof.

"Pull the vertices", one by one. □

The Upper Bound Theorem, 4D version

Theorem (Upper Bound Theorem, 4D version)

The face numbers of any 4*-polytope with* $f_0 = n$ *vertices satisfy*

$$\begin{aligned} f_1 &\leq \tfrac{1}{2}n(n-1) = \tbinom{n}{2}, \\ f_2 &\leq n(n-3), \\ f_3 &\leq \tfrac{1}{2}n(n-3), \end{aligned}$$

with equality for any neighborly 4*-polytope, for which any two vertices are adjacent.*

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with equality for any neighborly 4*-polytope, for which any two vertices are adjacent.*

Proof.

The inequality for f_1 is obvious.
We may assume that the polytope is simplicial.
Then $f_2 = 2f_3$.
Use the Euler-Poincaré equation. □

EXAMPLE: A Neighborly Polytope

EXAMPLE: $C_4(6) = \Delta_2 \oplus \Delta_2$ (sum of two triangles)

Neighborly 4-Polytopes Exist

Definition (Curve of order d)

A curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^d$ has *order d* if any hyperplane hits it in at most d points.

Neighborly 4-Polytopes Exist

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A curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^d$ has *order d* if any hyperplane hits it in at most d points.

Examples:

Any convex curve in the plane,
The *moment curve* $\gamma(t) = (t, t^2, \ldots, t^d)$,
The *binomial curve* $\delta(t) = (t, \binom{t}{2}, \ldots, \binom{t}{k})$,
The *Carathéodory curve* $c(t) = (\cos t, \sin t, \cos 2t, \sin 2t)$.

The Cyclic Polytopes

Definition (Cyclic Polytopes)

For $n > d \geq 2$, take any curve $x : \mathbb{R} \to \mathbb{R}^d$ of order d.
Then a d-dimensional *cyclic polytope* on n vertices is given by the convex hull of any n distinct points on the curve γ, that is

$$C_d(n) := \mathrm{conv}\{x(t_1), x(t_2), \ \ldots \ , x(t_n)\}$$

for real values $t_1 < t_2 < \cdots < t_n$.

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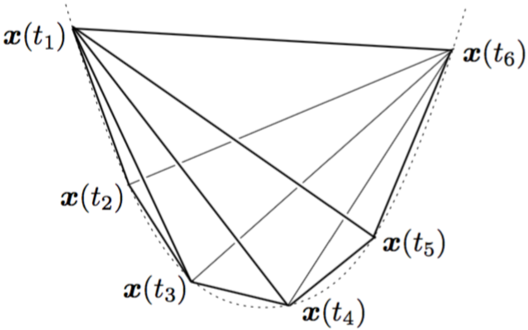

The Gale Evenness Criterion

Theorem (Gale Evenness Criterion)

Any cyclic d-polytope on n vertices $C_d(n)$ is simplicial.
Its vertices are given by those strings of d F's and $n - d$ O's,
for which the F's come in pairs,
except possibly at the beginning and the end.

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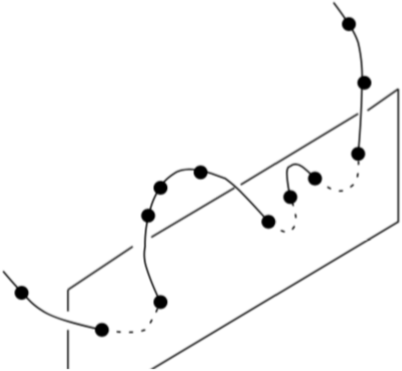
Proof.

By picture:
Example $d = 6$, $n = 12$
Facet $OFFOOOFFFFOO$



□

The Cyclic Polytopes are Neighborly

Corollary

The cyclic polytopes are neighborly:
Any $\lfloor d/2 \rfloor$ *vertices form a face.*
In particular, for $d \geq 4$*, the graph is complete.*

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Corollary

The cyclic polytopes are neighborly:
Any $\lfloor d/2 \rfloor$ vertices form a face.
In particular, for $d \geq 4$, the graph is complete.

Theorem (The Upper Bound Theorem, McMullen 1970)

The face numbers of any d-polytope P with $f_0 = n$ vertices satisfy

$$f_i(P) \leq f_i(C_d(n)).$$

Thus the cyclic polytopes simultaneously maximize all face numbers, among all d-polytopes with n-vertices.

OPEN: Cyclic/neighborly polytopes with small integer coordinates?

The f-vector problem

Describe the set $f(\mathcal{P}^d)$ of f-vectors of convex d-dimensional polytopes.

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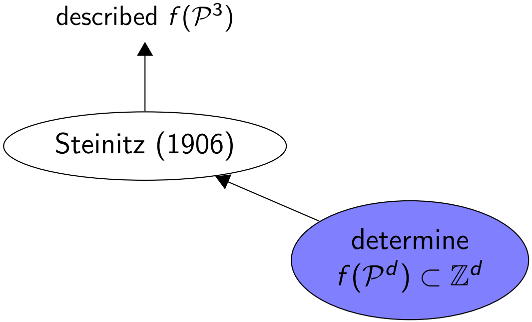
- find criteria that tell us whether a vector $(f_0, \ldots, f_{d-1})$ is an f-vector or not, and
- describe/characterize the set $f(\mathcal{P}^d)$.

For example,
OPEN PROBLEM: Is (1000, 10000, 10000, 1000) an f-vector of a 4-dimensional polytope?

The f-vector problem

determine
$f(\mathcal{P}^d) \subset \mathbb{Z}^d$

The f-vector problem



The f-vector problem

described $f(\mathcal{P}^3)$

Steinitz (1906)

determine $f(\mathcal{P}^d) \subset \mathbb{Z}^d$

$\dim f(\mathcal{P}^d) = d - 1$:
Euler's equation
is the only affine
equation for $f(\mathcal{P}^d)$

Grünbaum (1960s)

Projections of $f(\mathcal{P}^4)$
to two coordinates

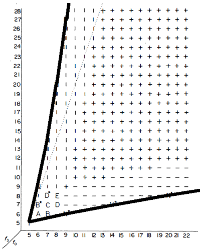


Figure 10.4.1

The f-vector problem

described $f(\mathcal{P}^3)$

Steinitz (1906)

characterizes $f(\mathcal{P}_s^d)$

g-Theorem (1980)

determine $f(\mathcal{P}^d) \subset \mathbb{Z}^d$

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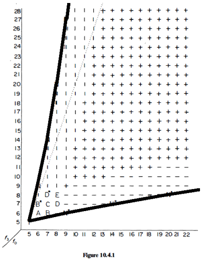
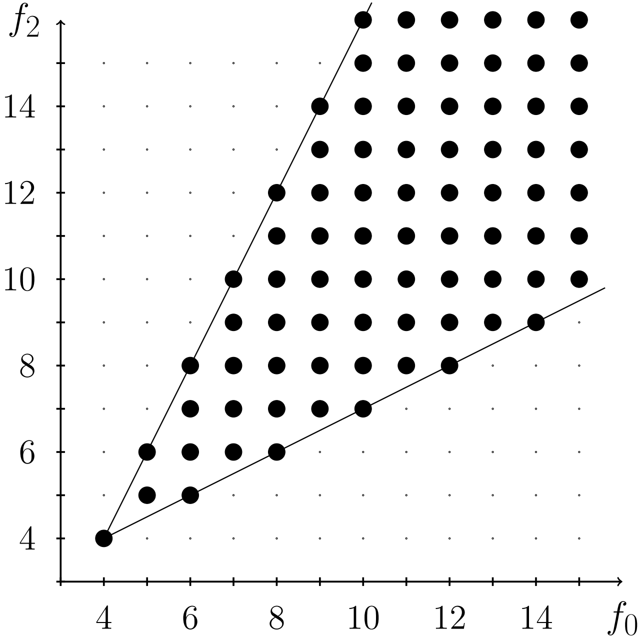


The f-vectors of 3-polytopes (Steinitz 1906)



The f-vectors of 3-polytopes (Steinitz 1906)

Theorem

The set of f-vectors of 3-dimensional polytopes is the set of all integer points in a 2-dimensional cone:

$$
\begin{aligned}
f(\mathcal{P}^3) \quad = \quad \{(f_0, f_1, f_2) \in \mathbb{Z}^3 : \quad & f_0 - f_1 + f_2 = 2, \\
& f_2 \leq 2f_0 - 4, \\
& f_0 \leq 2f_2 - 4 \ \}.
\end{aligned}
$$

The pairs (f_0, f_3) for 4-polytopes (Grünbaum 1967)

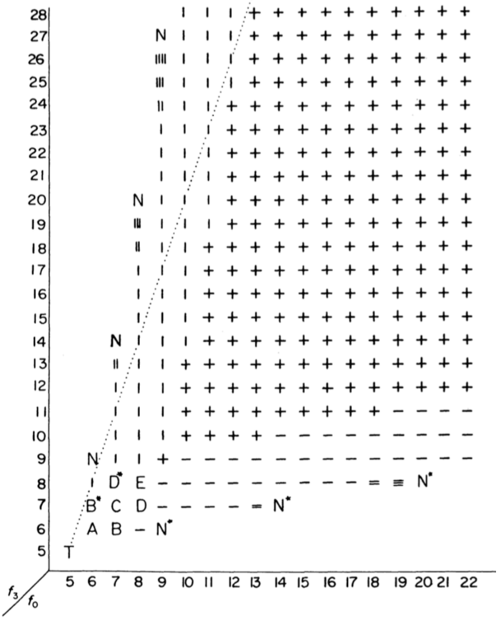
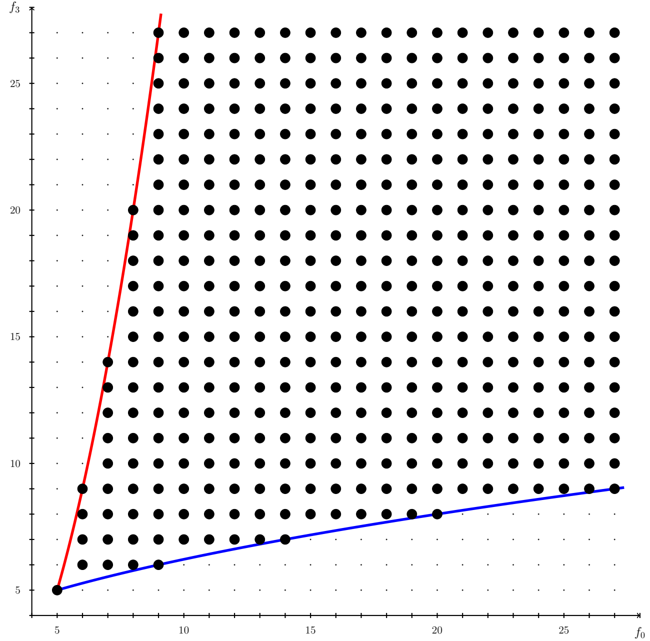


Figure 10.4.1

The pairs (f_0, f_3) for 4-polytopes (Grünbaum 1967)



The pairs (f_0, f_3) for 4-polytopes (Grünbaum 1967)

Theorem

The set of pairs (f_0, f_3) of vertex and facet numbers of 4-dimensional polytopes is the set of all integer points between two parabolas:

$$\begin{aligned}\pi_{03} f(\mathcal{P}^4) \quad = \quad \{(f_0, f_3) \in \mathbb{Z}^2 : \quad & f_3 \leq \tfrac{1}{2} f_0(f_0 - 3), \\ & f_0 \leq \tfrac{1}{2} f_3(f_3 - 3), \\ & f_0 + f_3 \geq 10 \ \}.\end{aligned}$$

The pairs (f_0, f_1) for 4-polytopes (Grünbaum 1967)

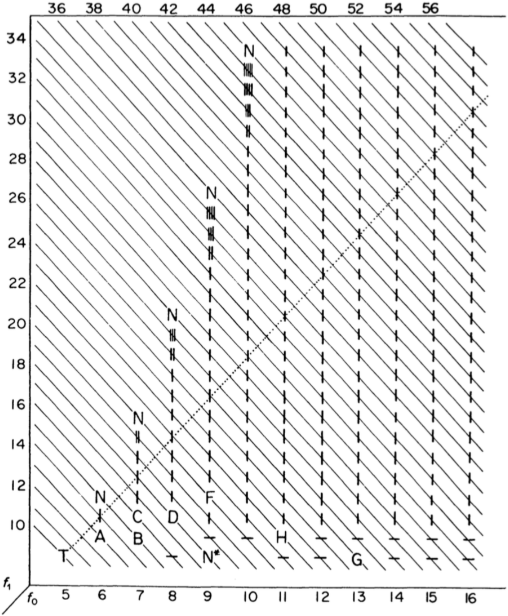
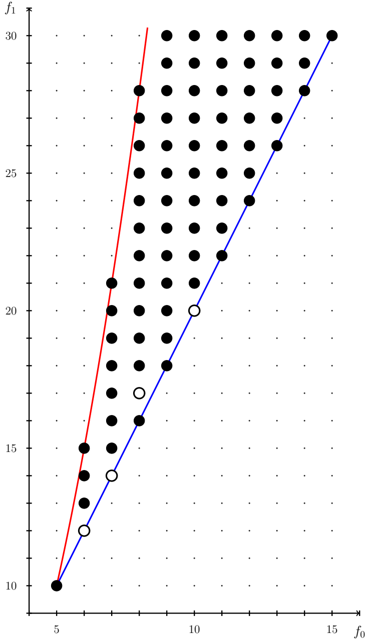


Figure 10.4.3

The pairs (f_0, f_1) for 4-polytopes (Grünbaum 1967)



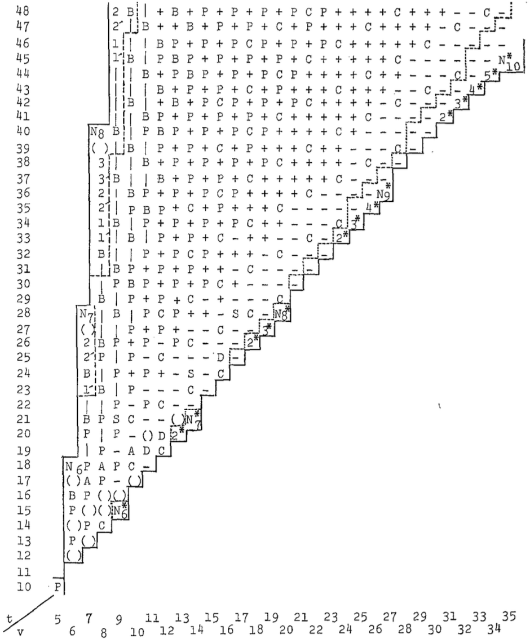
The pairs (f_0, f_1) for 4-polytopes (Grünbaum 1967)

Theorem

The set of pairs (f_0, f_1) of vertex and edge numbers of 4-dimensional polytopes is the set of all integer points between a line and a parabola, with four exceptions:

$$\pi_{01} f(\mathcal{P}^4) \;=\; \{(f_0, f_1) \in \mathbb{Z}^2 \;\;:\;\; 10 \le 2f_0 \le f_1 \le \tfrac{1}{2} f_0(f_0 - 1)\,\} \setminus \{\,(6,12),\,(7,14),\,(8,17),\,(10,20)\,\}.$$

The pairs (f_0, f_2) for 4-polytopes (Barnette & Reay 1973)



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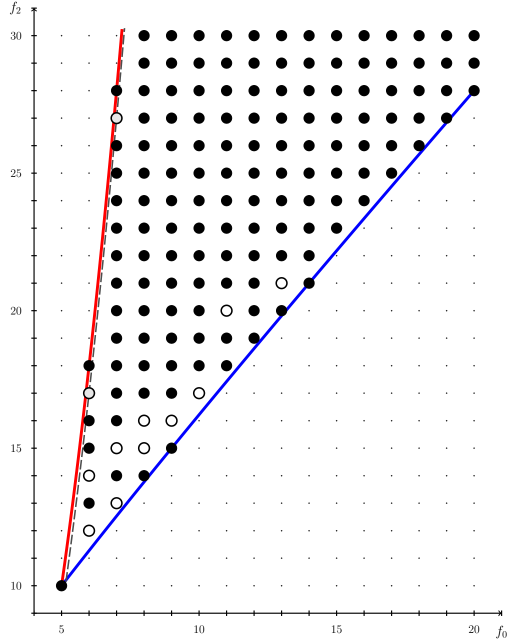


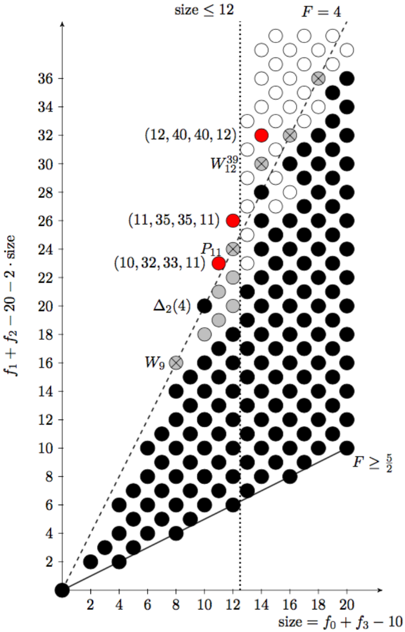
The pairs (f_0, f_2) for 4-polytopes (Barnette & Reay 1973)

Theorem

The set of pairs (f_0, f_2) of 4-dimensional polytopes is the set of all integer points between two parabolas, except for the integer points on an exceptional parabola, and ten more exceptional points:

$$
\begin{aligned}
\pi_{02} f(\mathcal{P}^4) \;=\; \big\{(f_0, f_2) \in \mathbb{Z}^2 \;:\; & 5 \le f_0, \\
& f_0 + \tfrac{3}{2} + \tfrac{1}{2}\sqrt{8f_0 + 9} \le f_2, \\
& f_2 \le f_0^2 - 3f_0, \\
& f_2 \ne f_0^2 - 3f_0 - 1 \big\} \\
\setminus \{\, & (6,12), (6,14), (7,13), (7,15), (8,15), \\
& (8,16), (9,16), (10,17), (11,20), (13,21) \,\}.
\end{aligned}
$$

The pairs $(f_0 + f_3, f_1 + f_2)$ (Brinkmann & Z. 2017)



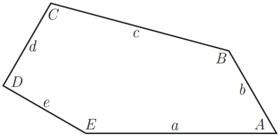
Lecture 4:
My Top Ten List of Examples

The List

1. A pentagon that tiles
2. A polyhedron that tiles
3. The pseudo-rhombicuboctahedron
4. A centrally-symmetric neighborly polytope
5. The 24-cell
6. The stellated 120-cell
7. The Klee–Walkup polytope
8. A neighborly cubical polytope
9. The Hansen polytope of a 4-path
10. The 4-simple, 4-simplicial Wythoff polytope

1. A Pentagon That Tiles

Casey Mann, Jennifer McLoud-Mann & David Von Derau (2015)



(a)

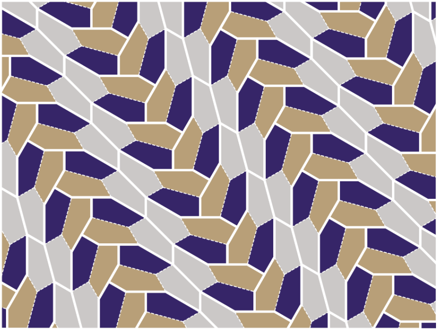
$A = 60°$ $\quad a = 1$

$B = 135°$ $\quad b = 1/2$

$C = 105°$ $\quad c = \dfrac{1}{\sqrt{2}(\sqrt{3}-1)}$

$D = 90°$ $\quad d = 1/2$

$E = 150°$ $\quad e = 1/2$



(b)

A 3-block transitive tiling by the Type 15 pentagon. The thick white lines outline the 3-block, and the colors of the tiles indicate the transitivity classes of pentagons.

Figure 10: The Type 15 pentagon

Image: Mann, McLoud-Mann & Von Derau (2015), `arXiv:1510.01186`

1. A Pentagon That Tiles

Theorem (Michaël Rao, 2017)

All pentagons that tile the plane are given by the list of 15 known families.

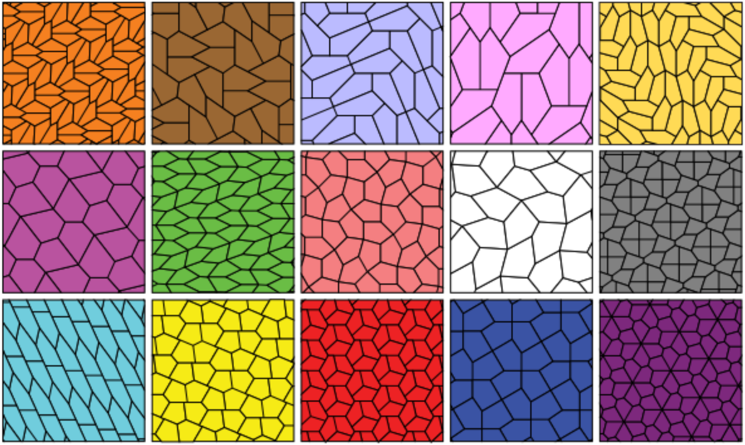


Image: Wikipedia

2. A Polyhedron That Tiles

$f(E) = (38, 106, 70)$ (Engel 1981)

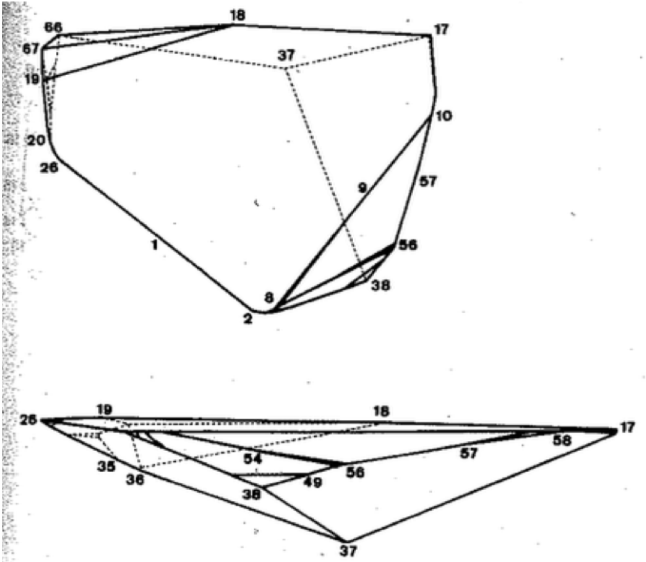


Abb. 4. Zwei Ansichten eines Exemplars des Polyedertyps 38.70-1

2. A Polyhedron That Tiles

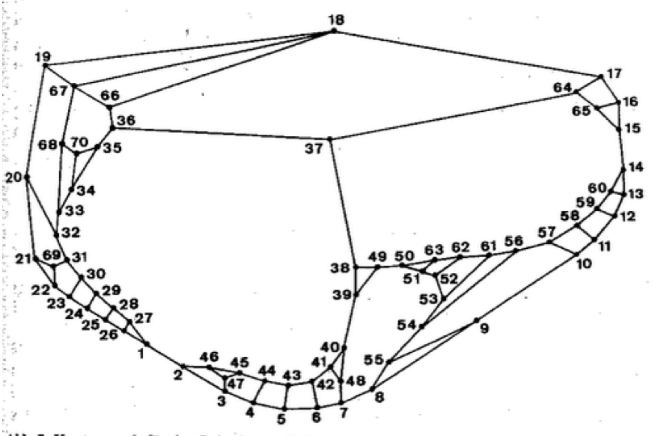


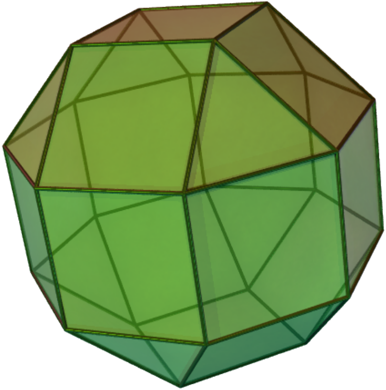
Abb. 5. Kantengraph für den Polyedertyp 38.70-1

2. A Polyhedron That Tiles

OPEN PROBLEMS:

- How many vertices/facets can polyhedron have that tiles?
- Is the maximum finite at all, in any dimension?

3. The Pseudo-Rhombicuboctahedron



... also known as "Miller solid" or "Johnson body J_{37}"

... first appears in paper by Sommerville (1906)

3. The Pseudo-Rhombicuboctahedron

OPEN PROBLEM:
Complete and correct write-up of the classification of the Archimedian solids, including
— combinatorial classification
— existence
— uniqueness

4. A Centrally-symmetric Neighborly 4-polytope

$P := \mathrm{conv}\{\Delta_4 \cup (-\Delta_4)\}$

$f = (10, 40, 60, 30)$, simplicial, centrally symmetric neighborly!

Its boundary arises as (2×5)-multi-chessboard complex

Recent discovery (Engström, A. Haase, Stump, Z.):
not vertex-decomposable!

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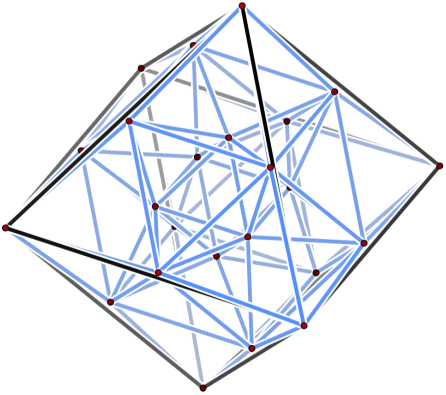
Recent discovery (Engström, A. Haase, Stump, Z.):
not vertex-decomposable!

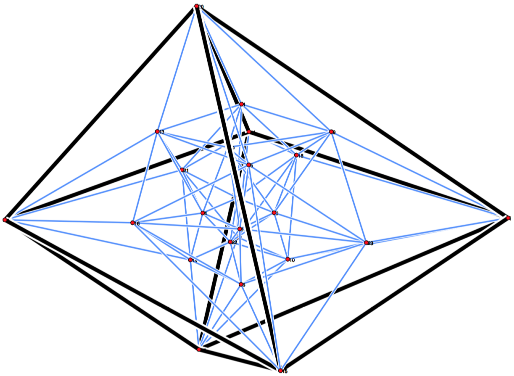
OPEN PROBLEM:
"How neighborly can a centrally-symmetric polytope be?"

5. The 24-Cell



(a) Schlegel diagram of the regular 24-cell.

5. The 24-Cell



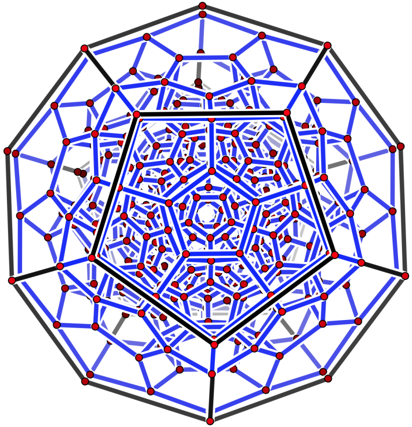
(b) Schlegel diagram of a polytope in the family of Table 3.5: $a_1, b_1 = \frac{1}{2}$.

Paffenholz (2005)

OPEN PROBLEM:

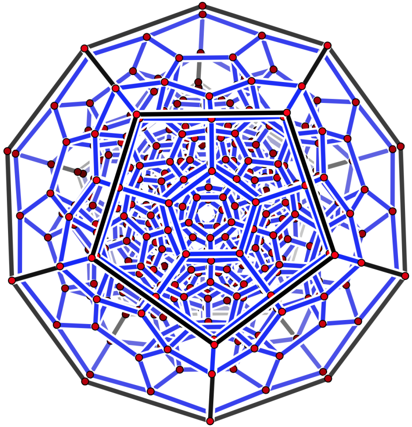
How does the realization space of the 24-cell look like? What's its dimension? Is it a manifold? [Rastanawi 2018]

6. The Stellated 120-Cell



f-vector $f = (720, 5040, 5040, 720)$

6. The Stellated 120-Cell



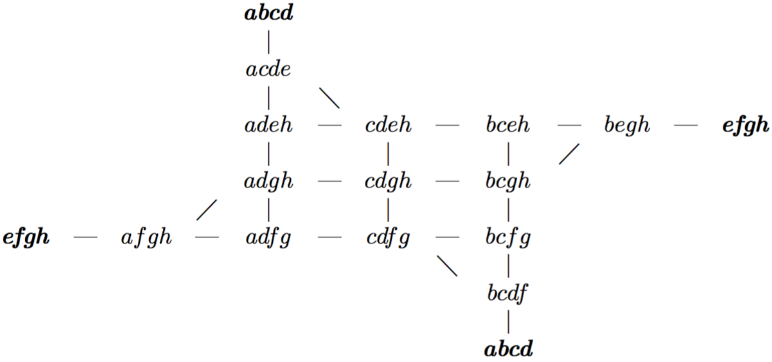
f-vector $f = (720, 5040, 5040, 720)$

OPEN PROBLEM:

- Does this polytope have rational coordinates?
- Is it projectively unique?

7. The Klee–Walkup Polytope

Klee & Walkup (1967) constructed a simple 4-polytope Q_4 with 9 facets and graph diameter 5 that is *tight* for the Hirsch conjecture. Its dual has f-vector $(9, 36, 54, 27)$: dual-to-neighborly!

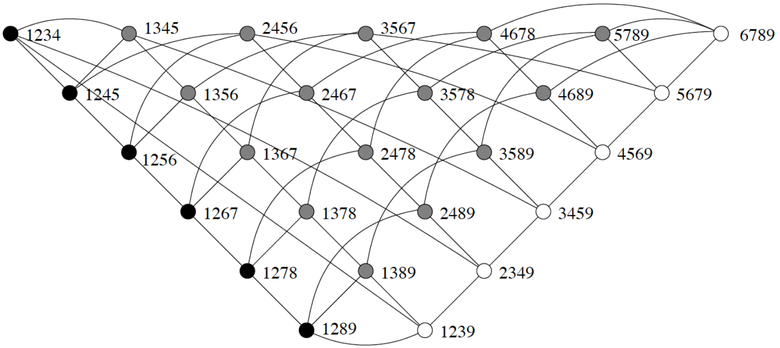


7. The Klee–Walkup Polytope

OPEN PROBLEMS:

- Identify this polytope in the classification neighborly 4-polytopes by Altshuler & Steinberg (1973).
- Is it the polytope that is extremal for the monotone upper bound problem, according to Pfeifle (2005)?
- The Hirsch conjecture for $d = 4$.

Pfeifle's $\widetilde{Q}_4$:

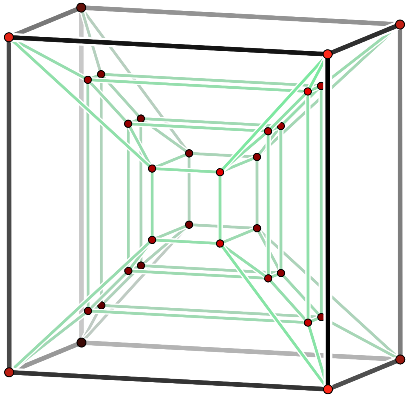


8. A Neighborly Cubical Polytope

A 4-polytope with the graph of the 5-cube:

$$\operatorname{conv}\{(2Q \times Q) \cup (Q \times 2Q)\}$$

for a square $Q = [-1, 1]^2$.



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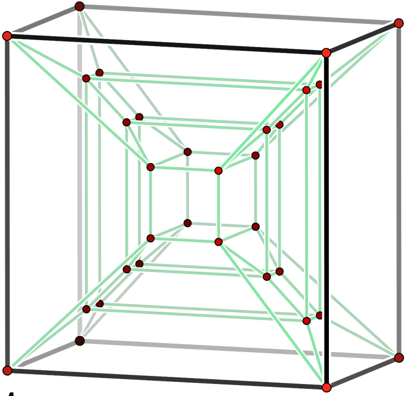

OPEN PROBLEM:

Are there centrally-symmetric neighborly n-cubical d-polytopes for all $n > d \geq 4$?

9. The Hansen Polytope of a 4-Path

Hansen's (1977) construction applied to the path P_4:

- ◦ centrally-symmetric
- ◦ dimension $d = 5$
- ◦ f-vector $(16, 64, 98, 64, 16)$

i.e. $3^d + 16$ non-empty faces.

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- ∘ centrally-symmetric
- ∘ dimension $d = 5$
- ∘ f-vector $(16, 64, 98, 64, 16)$

i.e. $3^d + 16$ non-empty faces.

OPEN PROBLEMS:
- Kalai's 3^d-conjecture:
  Does every c.s.-polytope have at least 3^d non-empty faces?
- Also Kalai:
  Does every c.s.-polytope have at least $2^d d!$ complete flags?
- Mahler's conjecture:
  Does every c.s. convex body have $V(B)V(B^*) \geq 4^d/d!$
- And what is the connection between these three conjectures?

(All of them are supposed to be tight at the Hanner polytopes.)

10. The 4-Simple, 4-Simplicial Wythoff polytope

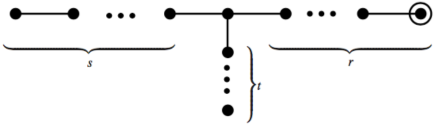

Figure 4.3: The general graph for Gosset–Elte polytopes.

- finite if
$$\frac{1}{r+1}+\frac{1}{s+1}+\frac{1}{t+1}>1$$
- dimension $d = r + s + t$
- $(r+2)$-simplicial and $(s+t-1)$-simple

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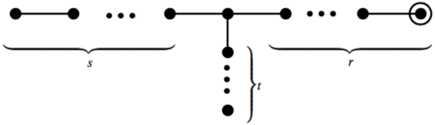

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"2_{41}" has dimension 8, is 4-simplicial 4-simple, f-vector
$(2160, 69120, 483840, 1209600, 1209600, 544320, 144960, 17520)$

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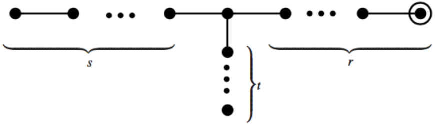

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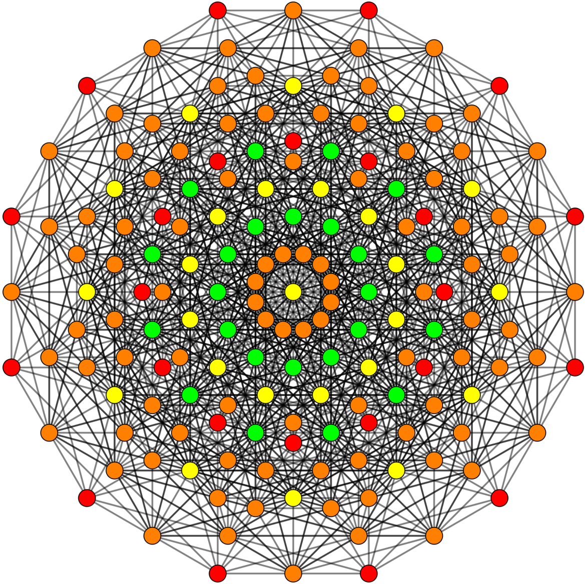
"2_{41}" has dimension 8, is 4-simplicial 4-simple, f-vector $(2160, 69120, 483840, 1209600, 1209600, 544320, 144960, 17520)$

OPEN PROBLEM:
Is there a 5-simple 5-simplicial polytope (other than the simplex)?

10. The 4-Simple, 4-Simplicial Wythoff Polytope



11. A 4-Polytope with Only Icosahedra as Facets

Is there a 4-polytope all whose facets are icosahedra?

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This follows from

$f_{03} = 12f_3$: Each facet has 12 vertices on average.
$f_{03} \geq 12f_0$: Each vertex is in at least 12 facets, so $f_0 \leq f_3$
$f_{02} = 3f_2$: Each 2-face has three vertices.