(Coimbra lecture)

Lectures II

Unconstrained Polynomial Optimization

$$p_* = \inf \{ p(x) : x \in \mathbb{R}^n \} = \sup \{ e : p(x) - e \geq 0 \}$$

$$p^{sos} := \sup \{ e : p(x) - e = sos \} \qquad p^{sos} \leq p_*$$

Example $p_* = \inf_{(x,y) \in \mathbb{R}^2} x^2 - xy^2 + y^4$

$$p^{sos} = \sup \{ e : x^2 - xy^2 + y^4 - e = sos \}$$

$$= \sup \{ e : x^2 - xy^2 + y^4 - e = [x]_2^T Q [x]_2 \}$$

$$[x]_2 := (1 \; x \; y \; x^2 \; xy \; y^2)$$

Let's try $x^2 - xy^2 + y^4 - e = (1 \; x \; y^2) \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{pmatrix} 1 \\ x \\ y^2 \end{pmatrix}$

$$= a + 2bx + 2cy^2 + dx^2 + 2e\, xy^2 + f y^4 \geq 0$$

$\Rightarrow$ $a = -e$ $\quad d = 1$

$b = 0$ $\quad 2e = -1 \Rightarrow e = -1/2$

$c = 0$ $\quad f = 1$

$$\begin{pmatrix} -e & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix} \succeq 0 \Rightarrow -e \geq 0 \Rightarrow e \leq 0$$

$$\sup \left\{ e : \begin{pmatrix} -e & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix} \succeq 0 \right\} = 0$$

$$\therefore \; p^{sos'} = 0 \leq p_* \leq p(0,0) = 0 \qquad \Rightarrow p_* = p^{sos} = 0$$

We could decompose p as a sos but not required to find p_*. Enough to find $Q \succeq 0$

Q: When is $p^{sos} = p_*$? Else how bad is the p^{sos} bd?

I) Nonnegative and SOS polynomials (Unconstrained POPs)

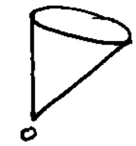
Defs: $p(x) \in \mathbb{R}[x]$ is nonnegative if $p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$

Recall $p \geq 0 \Rightarrow \deg p = 2d$, $\tilde{p} \geq 0$ $\therefore$ enough to study nonneg form

$\mathbb{R}[x]_{=2d} = \{$ forms of deg $2d$ in $\mathbb{R}[x]\}$ - vector space

Define $\mathcal{P}_{n,=2d} := \{ p \in \mathbb{R}[x]_{2d} : p(x) \geq 0 \quad \forall x \in \mathbb{R}^n \}$

Check: $\mathcal{P}_{n,=2d}$ is a convex cone in $\mathbb{R}[x]_{=2d}$



(Similarly $\mathcal{P}_{n,2d}$ is a convex cone in $\mathbb{R}[x]_{2d}$)

(pointed) $f, -f$ both not in (full-dim!) - small perturbations keep >0

Define $\Sigma_{n,=2d} := \{ p \in \mathbb{R}[x]_{=2d} : p = \sum q_i^2 \quad q_i \in \mathbb{R}[x]_{=d} \}$

full-dim! pointed convex cone in $\mathbb{R}[x]_{=2d}$

$$\boxed{\Sigma_{n,=2d} \subseteq \mathcal{P}_{n,=2d}} \qquad \boxed{\Sigma_{n,2d} \subseteq \mathcal{P}_{n,2d}}$$

Hilbert (1888): $\mathcal{P}_{n,=2d} = \Sigma_{n,=2d} \iff (n, 2d)$ is one of the foll:

1. $n = 2$ (all univariate nonneg polys)
2. $2d = 2$ (all quadratic (non homog) nonneg polys)
3. $n = 3$, $2d = 4$ (bivariate, deg 4 nonneg polys)

In all other cases $\exists$ nonnegative forms that are not sos i.e $\Sigma_{n,=2d} \subsetneq \mathcal{P}_{n,=2d}$.

Proof of 2: $\mathcal{P}_{n,=2} = \Sigma_{n,=2}$

$p \in \mathcal{P}_{n,=2} \Rightarrow p$ quadratic form in n vars $\Rightarrow p = x^T M x$
By the Spectral Thm for sym matrices, $M \in S^n$

$$M = U D U^T \qquad D_{ii} = \lambda_i(M)$$

$$\therefore p = x^T U D U^T x = (U^T x)^T D (U^T x) = \sum_{i=1}^n D_{ii} (U^T x)_i^2 \geq 0 \; \forall x$$

$$\Leftrightarrow D_{ii} \geq 0 \;\forall i \Leftrightarrow M \succeq 0 \Leftrightarrow M = BB^T$$

$$\Leftrightarrow p = x^T B B^T x = (B^T x)^T (B^T x) = \sum (B^T x)_i^2 \in \Sigma_{n,=2}$$

Corollary: Can minimize any quadratic poly in poly time via SDP. Let $q(x) \in \mathbb{R}[x]_2$

$$\inf_{x \in \mathbb{R}^n} q(x) = \sup\{e : \underbrace{q(x) - e \geq 0}_{\in \mathbb{R}[x]_2}\}$$

$q(x) - e^* = $ sos by Hilbert's thm (solve by SDP)

$$\therefore q^{sos} = q_*$$

In general, in all the Hilbert cases, $\mathcal{P}_{n,2d}$ has a nice description since it equals $\Sigma_{n,2d}$.
$\Sigma_{n,2d}$ is the projection of $S^{\binom{n+d}{d}}$ (projected spectrahedron)
For other cases, such nice descriptions don't exist.

Recall

$$\inf_{(x,y) \in \mathbb{R}^2} \{x^2 - xy^2 + y^4\} = \sup\{e : \underbrace{x^2 - xy^2 + y^4 - e \geq 0}_{n=2,\; 2d=4}\}$$

$$\therefore x^2 - xy^2 + y^4 - e^* = \text{sos}$$

$\Rightarrow p^{sos} = p_*$ and everything is ok.

So we need to understand what happens for $(n, 2d)$ outside the Hilbert cases.

Motzkin polynomial $\in \mathcal{P}_{3,=6} \setminus \Sigma_{3,=6}$ (4)

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

$M(x,y,z) \geq 0$: $M \geq 0 \iff \underbrace{x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \geq 0}_{\text{dehomogenization of } M}$

$\iff x^2y^2(x^2+y^2-3) + 1 \geq 0$

If $x^2+y^2 \geq 3$ then $x^2y^2(x^2+y^2-3) + 1 \geq 0 \quad \forall\, x,y$

Else $3 - x^2 - y^2 \geq 0$. Set $z^2 = 3 - x^2 - y^2$

By AM/GM inequality $\dfrac{x^2+y^2+z^2}{3} \geq \sqrt[3]{x^2y^2z^2}$

i.e. $1 \geq \sqrt[3]{x^2y^2(3-x^2-y^2)} \Rightarrow 1 + x^2y^2(x^2+y^2-3) \geq 0$. ∎

(Easier to see if you use the homog form - AM/GM inequality)

(*) $M(x,y,z)$ is not sos (exercise)

$\inf\limits_{(x,y,z)} M(x,y,z) = 0$ but the sos rel^n gives $p^{sos} = -\infty$.

$M(x,y)$ minimized at $(\pm 1, \pm 1)$ but $M(x,y) \neq$ sos

$\therefore$ What can we do about polynomials like this?

i.e $\inf\limits_{x \in \mathbb{R}^n} p = \sup\limits_{\text{s.t } p - e \geq 0} e$, $p - e^*$ not sos. ~~or $p_* > p^{sos}$~~

Idea: Use multipliers: Suppose $q(x)\,p(x)$ is sos where $q \geq 0$ then $p \geq 0 \rightsquigarrow$ yields a certificate for nonneg of p.

eg $(x^2+y^2)\,M(x,y) = y^2(1-x^2)^2 + x^2(1-y^2)^2 + x^2y^2(x^2+y^2-2)^2$

$\therefore$ If we replace $\sup\{e : M(x,y) - e \text{ sos}\}$ by

$\sup\{e : (x^2+y^2)\,M(x,y) - e \text{ sos}\}$ then get $\sup e = 0$

[Q] How do we choose multipliers?

[Q] How strong a certificate of nonnegativity can we get by using multipliers?

eg of a multiplier $q(x) = \sum q_i(x)^2$ (sos)

Then $p(x)\,q(x) = \sum s_j(x)^2 \iff p(x) = \frac{\sum s_j^2}{q} = \frac{(\sum s_j^2)\,q}{q \cdot q}$

$= \frac{(\sum s_j^2)(\sum q_i^2)}{q^2} = \frac{\sum\sum (s_j q_i)^2}{q^2} = \sum\sum \left(\frac{s_j q_i}{q}\right)^2$

i.e p is a sos of rational functions.

$\therefore$ Q: When is a nonneg poly a sos of rational fcns?

Hilbert's 17th problem: Is every nonnegative polynomial a sos of rational fcns?

Yes Artin (1924)

$\therefore\ p \geq 0 \iff p = \sum \left(\frac{s_i}{q_i}\right)^2$ Clearing denominators

we see that using sos polys as multipliers provides a sos certificate for all nonneg polys.

If $p(x)$ is fixed, we can search for a multiplier of fixed degree by solving for:

$q(x)$ sos, $q(x)\,p(x)$ sos

i.e $q(x) = [x]_k^T \underbrace{Q}_{\succeq 0} [x]_k = \sum Q_{\alpha,\beta}\, x^{\alpha+\beta}$

$q(x)\,p(x) = \left(\sum Q_{\alpha,\beta}\, x^{\alpha+\beta}\right)\left(\sum p_\gamma x^\gamma\right) = [x]_d^T A\, [x]_d \quad A \succeq 0$

Equate coeffs on both sides to get linear eqns in the entries of Q & A. Require $\begin{bmatrix} Q & 0 \\ 0 & A \end{bmatrix} \succeq 0 \rightsquigarrow$ SDP.

<u>Reznick's Thm</u> (1995) Suppose $p(x)$ homogeneous & $p > 0$ on $\mathbb{R}^n \setminus \{0\}$. Then $\exists\ r \in \mathbb{N}$ s.t.

$$\left(\sum x_i^2\right)^r p \text{ is sos.}$$

<u>Ex</u> For Motzkin $M(x, y, z)$ $r = 1$ suffices. (Note even the Motzkin form is not strictly positive)

<u>Ex</u>: <u>Def</u>: $M \in S^n$ is <u>copositive</u> if $x^T M x \geq 0$ $\forall x \in \mathbb{R}^n_+$

$$\iff p_M := \sum_{i,j=1}^n M_{ij} x_i^2 x_j^2 \geq 0$$

Testing whether a matrix is copositive is NP-complete.

eg $M = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \end{bmatrix}$ $\quad p_M = \sum_{i,j=1}^5 M_{ij} x_i^2 x_j^2$ quartic in 5 vars

p_M not sos but $\left(\sum_{i=1}^5 x_i^2\right) p_M$ is sos.

$$\Rightarrow p_M \geq 0 \Rightarrow M \text{ copositive.}$$

Reznick's thm creates a hierarchy of sos rel$^{\underline{n}}$s for unconstrained poly optimization:

$$\inf_{x \in \mathbb{R}^n} p(x) = \sup_{\text{s.t. } p - e \geq 0} e = \sup_{\text{s.t. } p - e > 0} e$$

$$\overset{\text{relax}}{\rightsquigarrow} \sup_{\underbrace{\left(\sum x_i^2\right)^r (p - e) \text{ sos}}_{r^{\text{th}} \text{ rel}^{\underline{n}}}} e = p_r^{\text{sos-Reznick}} \leq p_*$$

~~∃ finite r s.t.~~

$$\lim_{r \to \infty} p_r^{\text{sos-Reznick}} = p_* \quad \Big\} \text{ by Reznick's Thm}$$

Back to Hilbert's theorem : $\Sigma_{n,2d} \subseteq \mathcal{P}_{n,2d}$

First cases where $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ are $\begin{matrix} n=4 & d=4 \\ n=3 & d=6 \end{matrix}$

How can we see that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$?

Hilbert's idea $\rightsquigarrow$ Blekherman (Georgia Tech)

Consider $S = \{(\pm 1, \pm 1, \pm 1, 1)\} \subset \mathbb{R}^8$

Define $\pi : \mathbb{R}[x]_{4,4} \longrightarrow \mathbb{R}^8 \quad f \longmapsto (f(s_1), \dots, f(s_8))$

Let $\mathcal{P}' = \pi(\mathcal{P}_{4,4}) \quad \Sigma' = \pi(\Sigma_{4,4})$ ETS $\mathcal{P}' \neq \Sigma'$

also closed convex cones

closed convex cones

Claim $\mathcal{P}' = \mathbb{R}^8_+$

Clearly $\mathcal{P}' \subseteq \mathbb{R}^8_+$ since $\mathcal{P}$ consists of nonneg polys.
By convexity, suffices to show $e_i \in \mathcal{P}'$
By symmetry $x_i \mapsto -x_i$ suffices to show $e_1 \in \mathcal{P}'$.

Ex 4.8 Consider $p = \sum_{i=1}^{4} x_i^4 + 2 \sum_{i \neq j \neq k} x_i^2 x_j x_k + 4 x_1 x_2 x_3 x_4$

- $p \geq 0$, - p vanishes on exactly 7 pts in S.

$\therefore \mathcal{P}' = \mathbb{R}^8_+$

Claim : $\Sigma' \subsetneq \mathbb{R}^8_+$; $e_i \notin \Sigma'$ (again ETS $e_1 \notin \Sigma'$)

Suppose $\exists$ $p = \sum q_j^2$ $q_j \in \mathbb{R}[x]_{4,2}$ s.t. $\pi(p) = e_1$

$\Rightarrow p(s_2) = p(s_3) = \cdots = p(s_8) = 0 \Rightarrow q_j(s_i) = 0 \ \forall\, i = 2 \dots 8 \ \forall j$

The values of a quadratic form in $\mathbb{R}[x]_{4,2}$ on S satisfy a unique linear relation:

$$\sum_{s_i \text{ w/ even } \# 1\text{s}} q(s_i) = \sum_{s_i \text{ w/ odd } \# 1\text{s}} q(s_i)$$

$\therefore q_j(s_1) = 0$ as well $\forall j \Rightarrow p(s_1) = 0$ $\lightning$.

In practise it's very hard to find nonnegative polynomials that are not sos. (eg Motzkin)
We need to understand $\mathcal{P}_{n,2d}$ & $\Sigma_{n,2d}$
or dually $\mathcal{P}^*_{n,2d}$ & $\Sigma^*_{n,2d}$ (dual cones)

<u>Recall</u>: $K \subseteq \mathbb{R}^n$ closed convex cone $\quad K^* := \{ y \in (\mathbb{R}^n)^* : \langle x, y \rangle \geq 0 \ \forall x \in K \}$ non-neg linear func'ls on K

$\hat{\mathcal{P}}_{n,2d} := \mathcal{P}_{n,2d} \cap H \qquad H = \{ p \mid \int_{S^{n-1}} p(x)\, \mu(dx) = 1 \}$ hyperplane

$\hat{\Sigma}_{n,2d} := \Sigma_{n,2d} \cap H$

μ → rotation inv prob meas on S^{n-1}

(<u>Blekherman 2006</u>) $\exists$ constants $c_1, c_2 > 0$ (dep on d) s.t. for n large enough

$$c_1 n^{(d/2-1)/2} \leq \left(\frac{\operatorname{vol} \hat{\mathcal{P}}_{n,2d}}{\operatorname{vol} \hat{\Sigma}_{n,2d}} \right)^{1/D} \leq c_2 n^{(d/2-1)/2} \qquad D = \binom{n+2d-1}{2d} - 1$$

~~Berg / Lasserre - Netzer If n is fixed but $d \to \infty$ then $\Sigma_{n,2d}$ is dense in $\mathcal{P}_{n,2d}$~~

$\exists$ significantly more nonneg polys than sos

Define $\quad \Theta_t := \sum_{k=0}^{t} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!} \qquad \deg \Theta_t = 2t$

Lasserre (2006) Let $f \in \mathbb{R}[x]$ be non-negative on $\mathbb{R}^n$.
For any $\varepsilon > 0$ $\exists$ $t_0 \in \mathbb{N}$ s.t
$f + \varepsilon \Theta_t$ is a sos $\forall$ $t \geq t_0$

(i.e for high enough degree t we can make $f + \varepsilon \Theta_t$ a sos)